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HOMOTOPIES USING CONFORMAL TRANSFORMATION WITH INVARIANT TOTAL NORMAL TWIST

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The aim of this paper is to detect a homotopy to a spherical curve with invariant total normal twist 0. Also, we use conformal transformation to prove a theorem that for any immersed closed C^3 -curve c in the Euclidean 3-space E^3 with vanishing total normal twist, there exists a Frenet curve \tilde{c} homotopic to c in an arbitrary neighborhood of c such that the total normal twist is invariant along the homotopy between c and \tilde{c} in that neighborhood.

1. Introduction.

Let $c : S^1 \longrightarrow E^3$ be a regular closed smooth curve (at least of class C^3) for the subsequent considerations. Parallel transfer of the normal plane along one period of the curve c with respect to the normal connection leads to a rotation of the normal plane which is characterized (up to integer multiples of 2π) by an oriented angle $\alpha(c)$ which we call the *total normal twist* of c . For Frenet curves this quantity is given by their total torsion up to integer multiples of 2π (see [1] and [6]).

It has been shown in [3] that the total normal twist of a closed curve is invariant under similarities (homotheties). In the work of B. Wegner [7] it has

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been shown that parallel sections of the normal bundle may remain parallel after renormalization, if the ambient space is subjected to a conformal transformation and consequently the local and uniform parallel ranks of immersions [2,5] into Euclidean 3-space E^3 are preserved under conformal transformations. Also, if the total normal twist of an immersed closed curve in E^3 is an integer multiple of 2π , then the same is true for any image of the curve under a conformal transformation of the ambient space.

2. A homotopy to spherical curve.

We begin with proving the existence of a homotopy in the Euclidean 3-space E^3 from a plane curve to a spherical one preserving the total normal twist.

Theorem 2.1. *Let c be a closed plane curve (at least of class C^3). Then there exist a curve c^* in $S^2(R)$ and a homotopy from c to c^* with total normal twist 0.*

Proof. Let ϕ be the stereographic projection from the north pole x_0 of the sphere $S^2(R)$ of radius R onto the tangent plane of the south pole $\mathbb{R}^2 \times \{0\}$ (see Fig. 1). Assume that the image of c is contained in the plane. Then the coordinates of the center of the sphere are $p_0 = (0, 0, R)$, and those of the north pole $x_0 = (0, 0, 2R) = 2p_0$.

The mapping ϕ takes $x \in S^2(R) - \{x_0\}$ into the intersection of the plane $\mathbb{R}^2 \times 0$ with the line that passes through x and x_0 . It is clear that $R = \|x - p_0\|$. Then we have

$$R = \|r\phi(x) + (1 - r)x_0 - p_0\|$$

for some $r \in [0, 1]$. This will be

$$R = \|r\phi(x) + (2(1 - r) - 1)p_0\|,$$

i.e.

$$(1) \quad R = \|r\phi(x) + (1 - 2r)p_0\|.$$

Equation (1) gives us

$$R^2 = r^2 \|\phi(x)\|^2 + (1 - 2r)^2 R^2 = r^2 \|\phi(x)\|^2 + R^2 + 4r^2 R^2 - 4r R^2,$$

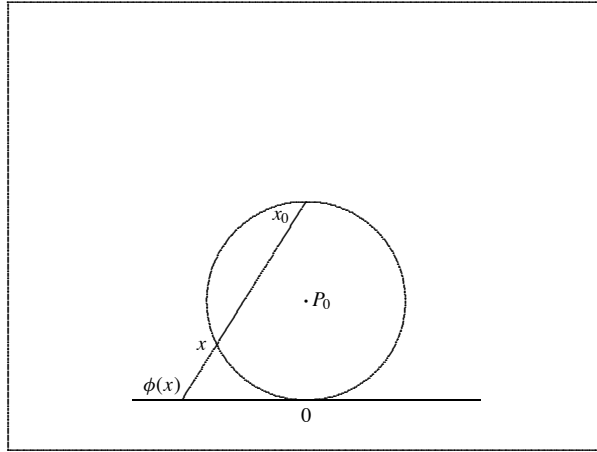


Figure 1.

(note that $\langle \phi(x), p_0 \rangle = 0$), i.e.

$$r(r\|\phi(x)\|^2 + 4R^2) - 4R^2 = 0.$$

The above equation gives us $r = 0$ or

$$(2) \quad r = \frac{4R^2}{\|\phi(x)\|^2 + 4R^2},$$

and

$$(3) \quad 1 - r = \frac{\|\phi(x)\|^2}{\|\phi(x)\|^2 + 4R^2}.$$

Substituting Equations (2), (3) in $x = r\phi(x) + (1 - r)x_0$ gives

$$(4) \quad x = \frac{4R^2}{\|\phi(x)\|^2 + 4R^2} \phi(x) + \frac{\|\phi(x)\|^2}{\|\phi(x)\|^2 + 4R^2} x_0.$$

Setting $R = \frac{1}{\lambda}$ and denoting more precisely with Φ_λ the stereographic projection belonging to the sphere with radius $\frac{1}{\lambda}$, Equation (4) will be

$$(5) \quad x = \frac{1}{\lambda^2 \|\Phi_\lambda(x)\|^2 + 4} \{4\Phi_\lambda(x) + \lambda^2 \|\Phi_\lambda(x)\|^2 x_0\}.$$

Hence let c be a closed curve in the plane $\mathbb{R}^2 \times \{0\}$ and consider this as the image of the curve $\Phi_\lambda^{-1} \circ c$ on $S^2(\frac{1}{\lambda})$. Using the formula (5) we can define a homotopy H as follows:

$$(6) \quad H(t, \lambda) = \frac{1}{\lambda^2 \|c(t)\|^2 + 4} \{4c(t) + \lambda^2 \|c(t)\|^2 x_0\}.$$

It is clear that $H(t, 0) = c(t)$ which has total normal twist 0 (because c is a plane curve). $\lambda = 1$ defines a spherical curve

$$c^*(t) = \frac{1}{\|c(t)\|^2 + 4} \{4c(t) + \|c(t)\|^2 x_0\}$$

lying in the sphere $S^2(1)$ with center $(0, 0, 1)$. For any $\lambda \in [0, 1]$, $H(t, \lambda)$ is a spherical curve lying in $S^2(\frac{1}{\lambda})$, and consequently it has total normal twist 0, and this proves the theorem. \square

Remark. At $\lambda = 1$, we have the sphere $S^2(1)$ of this family of ambient spheres, containing c^* . For decreasing values of λ from 1 to 0 in the interval $[0, 1]$ the corresponding sphere $S^2(\frac{1}{\lambda})$ begins to blow up and finally tends to a sphere passing through infinity (plane) at $\lambda = 0$ ($R = \infty$) where c was located.

Note that not all the notions given above in E^3 are straightforward in E^4 . Hence it is suitable to give some remarks about curves in E^4 before giving our main theorem.

Proposition 2.1. *Let $c : S^1 \longrightarrow E^4$ be a regular closed smooth curve in E^4 , then there exists a parallel section ξ of the normal bundle of c .*

Proof. Let l denote the length of c . According to the definition of normal holonomy map,

$$A : \nu_c(s) \subset E^3 \longrightarrow \nu_c(s + l) = \nu_c(s) \subset E^3$$

it is an orientation preserving linear isometry, where $\nu_c(s)$ is a normal vector space of c at s . Hence A has 1 as an eigenvalue of multiplicity 1 at least. Let

$$A(\xi_0) = \xi_0,$$

i.e. ξ_0 is an eigenvector of A in $\nu_c(s)$. This implies that ξ_0 can be extended by parallel transfer in the normal bundle to a globally defined vector field ξ . \square

To define the total normal twist of c in E^4 let $\{T, N_1, N_2, N_3\}$ be an orthonormal frame field along c , T denoting the field of unit tangents. As has

been shown in Proposition 2.1, we may choose N_1 as a parallel section of the normal bundle of c . Hence starting parallel transfer of $N_i(0)$, $i = 1, 2, 3$, in $c(0)$ will leave $N_1(0)$ invariant after one period, and it will rotate $N_2(0)$ and $N_3(0)$ by an angle $\alpha(c)$ in the $(N_2(0), N_3(0))$ -plane. This may be taken as the total normal twist α of c in E^4 .

Under these assumptions the calculation of α in terms of the given frame field was done in our work [3], and it will takes the following formula:

$$\alpha(c) = \int_0^1 \omega_{32}(s) ds,$$

where $\omega_{32} = \langle \nabla_T N_3, N_2 \rangle$.

A spherical curve in E^3 has total normal twist zero. By using the above formula it is easy to extend this fact to spherical curves in E^4 . For more details see [4], [6].

Theorem 2.2. *Let $c : S^1 \longrightarrow E^3$ be an immersion of class C^3 in the Euclidean 3-space E^3 such that $\alpha(c) = 0$. Then for each $\delta > 0$, there exist a Frenet curve \tilde{c} with $\|\tilde{c} - c\|_3 < \delta$ and a homotopy H_λ between c and \tilde{c} , $\lambda \in [0, 1]$, with $\|H_\lambda - c\|_3 < \delta$ such that $\alpha(H_\lambda) = 0$ for all $\lambda \in [0, 1]$.*

Proof. Let $c : S^1 \longrightarrow E^3$ be a regular smooth curve of class C^3 parametrized by arc length in the Euclidean space E^3 . Let

$$\phi : S^3(1) - \{x_0\} \longrightarrow E^3$$

be the stereographic projection from the north pole x_0 into the hyperplane $E^3 \subset E^4$. Then the spherical curve

$$\hat{c} = \phi^{-1} \circ c : S^1 \longrightarrow S^3(1) \subset E^4$$

is of class C^3 . Since ϕ is a conformal mapping, tangents of c are sent to tangents of \hat{c} , and osculating circles of c are sent to osculating circles of \hat{c} . It is clear that \hat{c} is a Frenet curve (\hat{c} has non-zero curvature).

Defining a mapping

$$\Psi : S^1 \times S^1 \longrightarrow S^3(1)$$

by

$$\Psi(s, \theta) = m(s) + r(s)(N(s) \cos \theta + T(s) \sin \theta),$$

where $m(s)$ is the center of curvature of \hat{c} at s , $r(s)$ is the radius of curvature of \hat{c} at s . Then Ψ is differentiable of class C^1 . Since $\dim(S^1 \times S^1) = 2 < \dim S^3(1) = 3$, we have that $\Psi(S^1 \times S^1)$ is nowhere dense in $S^3(1)$, i.e.

$S^3(1) - \Psi(S^1 \times S^1)$ is dense in $S^3(1)$. Hence for given $\varepsilon > 0$, there exists $\hat{x} \in S^3(1) - (\Psi(S^1 \times S^1) \cup \{x_0\})$ such that

$$\|\hat{x} - x_0\| < \varepsilon.$$

Rotating \hat{c} into \hat{c}_1 in $S^3(1)$, $\hat{c}_1 = R_1 \circ \hat{c}$, with the unique rotation R_1 mapping \hat{x} to x_0 (\hat{x} has been brought into the position of the north pole). Applying the stereographic projection

$$\phi : S^3(1) - \{x_0\} \longrightarrow E^3$$

to the curve \hat{c}_1 . This yields a curve \tilde{c}_1 in E^3 , $\phi \circ \hat{c}_1 = \tilde{c}_1$. Since ϕ is C^∞ , there exists for given $\delta > 0$ and c an $\varepsilon > 0$ such that with the notations above

$$\|\phi \circ \hat{c} - \phi \circ \hat{c}_1\|_3 < \delta,$$

i.e.

$$\|c - \tilde{c}_1\|_3 < \delta.$$

But we have that $\hat{x} \notin \Psi(S^1 \times S^1)$, which implies that \tilde{c}_1 is a Frenet curve in E^3 . The spherical curve $\hat{c}_1 \in S^3(1)$ has total normal twist 0 (integer multiple of 2π) as we have shown before and illustrated in [4], [6]. Also, since ϕ is a conformal mapping, then \tilde{c}_1 has total normal twist 0.

Now we are going to prove that there exists a homotopy between the curves c and \tilde{c}_1 in the δ -neighborhood of c preserving the total normal twist 0. Connecting \hat{x} with x_0 by a shortest geodesic path in $S^3(1)$. It is clear that for any

$$x_\lambda = \lambda \hat{x} + (1 - \lambda)x_0, \quad \lambda \in [0, 1],$$

the rotation R_λ mapping x_λ to x_0 yields a rotation of \hat{c} to \hat{c}_λ , i.e. $R_\lambda \circ \hat{c} = \hat{c}_\lambda$, where $\hat{c}_0 = \hat{c}$, such that

$$\|x_\lambda - x_0\| < \varepsilon.$$

Then applying ϕ to \hat{c}_λ gives a C^3 curve \tilde{c}_λ in the Euclidean space E^3 such that

$$\|c - \tilde{c}_\lambda\|_3 < \delta,$$

and $\alpha(\tilde{c}_\lambda) = 0$. Hence the homotopy

$$H : S^1 \times [0, 1] \longrightarrow E^3$$

between c and \tilde{c}_1 is given by

$$H(s, \lambda) = (\phi \circ R_\lambda \circ \phi^{-1})(c(s))$$

where R_λ is the rotation mapping of x_λ to the north pole x_0 and satisfies the conditions required.

It is easy to see that

$$H(s, 0) = (\phi \circ Id \circ \phi^{-1})(c(s)) = c(s),$$

$$H(s, 1) = (\phi \circ R_1)(\hat{c}(s)) = \phi(\hat{c}_1(s)) = \tilde{c}_1(s),$$

and $\alpha(H(., \lambda)) = 0$, because all the curves of the family $H(., \lambda)$ are images of spherical curves under conformal transformations. \square

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